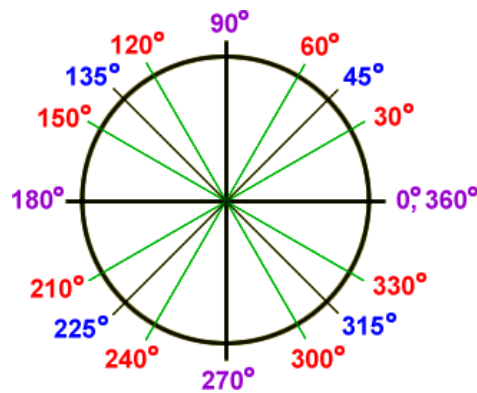# Math Boot Camp
Trigonometry
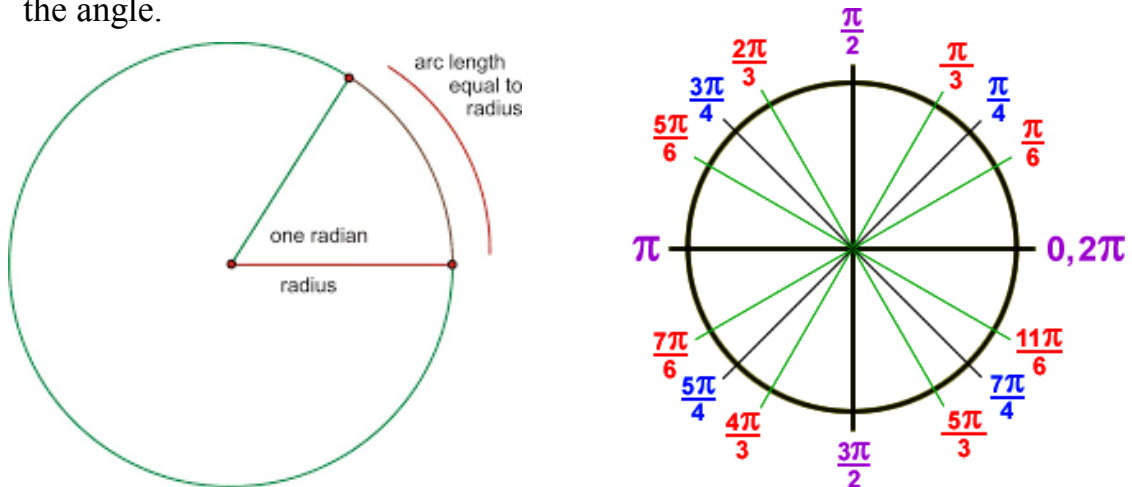
Angle measurement

Degrees

There are two commonly used units of measurement for angles. The more familiar unit of measurement is that of degrees. A circle is divided into 360 equal degrees, so that a right angle is 90°. We can extend this idea to include angles greater than 360 degrees or even negative angles.

Radians

Typically, the more useful measurement for angles is radians. For this measurement, consider a circle of radius 1 whose center is the vertex of the angle in question (this circle is often called the *Unit Circle*). Then the angle cuts off an arc of the circle, and the length of that arc is the radian measure of the angle.

To convert between degree measurement and radian measurement, note that the circumference of the entire circle is 2π, so it follows that $360°$ equals 2π radians.
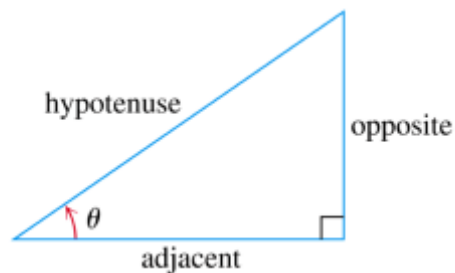
Hence,

$$180° = \pi \, radians.$$

Special Angles measured in degrees and radians					
Degrees:	0	30	45	60	90
Radians:	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{4}$	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$

Right-Triangle Trigonometry

Given the right triangle below, the following ratios are defined relative to the angle θ.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

Sine, Cosine and Tangent of Common Angles

In your calculus class, you will frequently need to evaluate a trigonometric function at a given angle. In the table below sine, cosine and tangent are calculated for the most common angles you will work with in calculus.

Values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for selected values of θ					
Degrees	0	30	45	60	90
θ (radians)	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{4}$	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$
$\sin\theta$	0	$\dfrac{1}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\dfrac{\sqrt{3}}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{1}{2}$	0
$\tan\theta$	0	$\dfrac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$Und.$

Note: The information in the table above is the **minimum** expectation regarding trigonometric functions of given angles for calculus students. Once we have these memorized we can easily answer questions like:
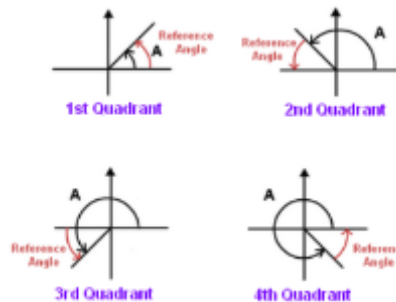
Example:

Evaluate $\sin\dfrac{\pi}{3}$

Solution:

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

Since all of the angles represented in the table are in the first quadrant, we must develop a method for evaluating these *special angles* as they arise in other quadrants. This is done by first recognizing that $\sin\theta$, $\cos\theta$ and $\tan\theta$ are all positive in the first quadrant. In the second quadrant only $\sin\theta$ is positive. In the third quadrant only $\tan\theta$ is positive and $\cos\theta$ is the only positive trigonometric function (of these three) in the fourth quadrant (see graphs below). An easy way to remember this is the mnemonic "All Students Take Calculus." Now just evaluate the trigonometric function of the *reference angle* (see below) and apply the correct sign.
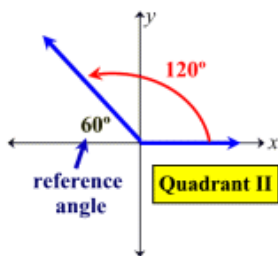
Students	All
sine	all
Take	Calculus
tangent	cosine

1st Quadrant   2nd Quadrant

3rd Quadrant   4th Quadrant

Examples

1) Evaluate $\cos 120°$

Solution: The reference angle for $120°$ is $60°$ (see below) and $\cos 60° = \frac{1}{2}$. However, $120°$ is a second quadrant angle and $\cos\theta$ is negative in the second quadrant.

120°

60°

reference angle   Quadrant II

Therefore, $\cos 120° = -\frac{1}{2}$

2) Evaluate $\sin\frac{5\pi}{6}$

Solution: The reference angle for $\frac{5\pi}{6}$ is $\frac{\pi}{6}$ and $\sin\frac{\pi}{6} = \frac{1}{2}$. However, $\frac{5\pi}{6}$ is a fourth quadrant angle, so $\sin\frac{5\pi}{6}$ must be negative.

Therefore, $\sin\frac{5\pi}{6} = -\frac{1}{2}$.

3) Evaluate $\tan\frac{5\pi}{4}$

Solution: The reference angle for $\frac{5\pi}{4}$ is $\frac{\pi}{4}$ and $\tan\frac{\pi}{4} = 1$. Since $\frac{5\pi}{4}$ is a third quadrant angle and $\tan\theta$ is positive in the third quadrant there is no sign change.
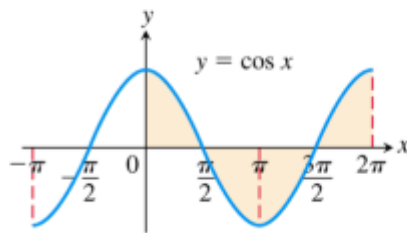
Therefore, $\tan\frac{5\pi}{4} = 1$.

Practice

1) Evaluate $\sin\left(\frac{3\pi}{4}\right)$
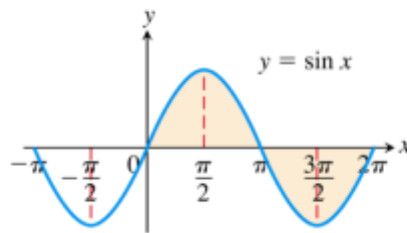
2) Evaluate $\cos 135°$
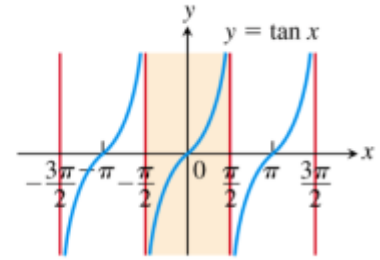
3) Evaluate $\tan\left(-\frac{\pi}{3}\right)$

Graphs

Domain: $(-\infty, \infty)$

Range: $[-1, 1]$

Domain: $(-\infty, \infty)$

Range: $[-1, 1]$

Domain:
{all x in $\mathbb{R}$: $x \neq$ odd multiple of $\frac{\pi}{2}$}

Range: $(-\infty, \infty)$

Remarks: While the ability to create the graphs of all the trigonometric functions from memory would be useful, it's more practical to focus on their general shapes, domains, ranges and asymptotes. Furthermore, it's most important to be familiar with the sine and cosine functions. The fact that the outputs, y-values, of sine and cosine are between -1 and 1 is a feature we leverage quite frequently in calculus and should be noted here.

Example: In calculus we will often want to sandwich or pinch one function between two other functions. Find two functions that sandwich $f(x) = \frac{\cos x}{x}$
Since $-1 \leq \cos x \leq 1$, it follows that
$$-\frac{1}{x} \leq \frac{\cos x}{x} \leq \frac{1}{x}$$

Inverse Trigonometric Functions

An inverse function "reverses" another function. In general, if $f(x) = y$ then the inverse of f, written f^{-1}, takes y as the input and produces x, i.e., $f^{-1}(y) = x$.

For trigonometric functions (sine, cosine, tangent, etc.) the inputs are angles, θ, and the outputs are real numbers. Thus, inverse trigonometric functions

$(\sin^{-1}, \cos^{-1}, \tan^{-1},$ etc.) have real number inputs and their outputs are angles (see diagram below).
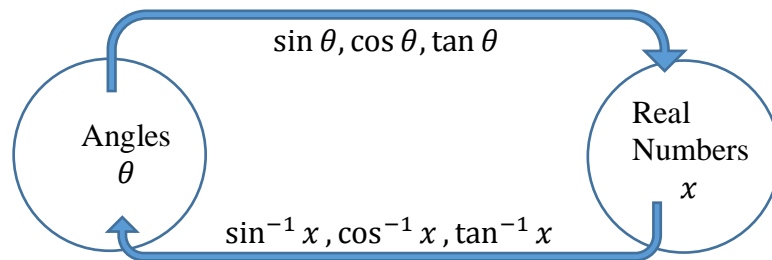
Note: $\sin^{-1} x$ is the angle whose sine is x. Thus, for every x in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\sin(\sin^{-1} x) = x$ and $\sin^{-1}(\sin x) = x$.

Example

Solve the trigonometric equation
$$\sin \theta - 1 = 0, \quad 0 \le \theta < 2\pi$$

Solution:

$\sin \theta = 1$

$\theta = \frac{\pi}{2}$ (i.e., $\frac{\pi}{4}$ or $90°$ is the angle between 0 and 2π whose sine is 1)

Practice

1) Solve the trigonometric equation $\cos x + \frac{\sqrt{2}}{2} = 0, \ 0 \le x < \pi$

Commonly Used Identities in Calculus

Pythagorean Identity (#1 most common, must know)

$$\cos^2 \theta + \sin^2 \theta = 1.$$

Others that show up every now and again:

Half-Angle Formulas (or Power Reducing Formulas)
Sum and Difference Formulas
Double-Angle Formulas

Example:

In calculus, you will be asked to calculate the "derivative" of a function like
$$f(x) = \frac{\sin^2 x}{1 - \cos^2 x}$$

Many students will simply apply the "Quotient Rule" and calculate the derivative (a doable, but unpleasant task). However, if we simply notice that $\sin^2 x + \cos^2 x = 1$ implies $1 - \cos^2 x = \sin^2 x$. We then have

$$f(x) = \frac{\sin^2 x}{1 - \cos^2 x} = \frac{\sin^2 x}{\sin^2 x} = 1.$$

Early in calculus you'll learn the derivative of a constant (in this case 1) equals zero. Thus, the derivative of $f(x) = 0$.

Practice

1) Use the Pythagorean Identity to show that
$$\cot^2 x + 1 = \csc^2 x$$