Math Boot Camp

Exponentials and Logarithms

In this section, your focus should be to master the patterns in equations (1)-(5) below. This means recognizing the expression on the left and replacing it with the expression on the right (left to right) as well as going from right to left. But first, a short introduction.

Exponents

In an expression like a^b, b is called the exponent and a is called the base.

If b is an integer, a^b is shorthand for $a \cdot a \cdot a \cdots a$, where a is multiplied by itself b times. If b is a fraction, the meaning is similar. For example, $8^{1/3}$ is the number that, when multiplied by itself 3 times produces 8. In other words, $8^{1/3} = 2$.

Rules of Exponents: If $a > 0$ and $b > 0$, the following rules hold for all real numbers x and y.

	Rule	**Example**
(1)	$a^x a^y = a^{x+y}$	$2^3 2^2 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32$
(2)	$\frac{a^x}{a^y} = a^{x-y}$	$\frac{2^3}{2^2} = \frac{2 \cdot 2 \cdot 2}{2 \cdot 2} = 2^{3-2} = 2$
(3)	$(a^x)^y = (a^y)^x = a^{xy}$	$(2^3)^2 = (2 \cdot 2 \cdot 2)^2$ $= (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2)$ $= 2^{3 \cdot 2} = 2^6 = 64$
(4)	$a^x b^x = (ab)^x$	$2^3 5^3 = (2 \cdot 5)^3 = 10^3 = 1000$
(5)	$\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$	$\frac{2^3}{5^3} = \left(\frac{2}{5}\right)^3$

The rules above are can be used in both directions. Whenever you have an expression that matches the left side of the rule, you can replace it with the expression on the right and vice versa.

Calculus Example

$$\begin{aligned} \int e^{x+2}dx &= \int e^x e^2 dx \\ &= e^2 \int e^x dx \end{aligned}$$

Here, rule (1) is applied to convert the integral into a form that you'll learn is easy to integrate.

Power Functions vs Exponential Functions

You are probably familiar with a function like $f(x) = x^5$. For this function, the variable (the input of the function) is the base, x, and the exponent 5 is constant (is not an input and doesn't change). Functions like this are called **power functions**.

Functions like $f(x) = 3^x$ are also commonly encountered in calculus. Here, the variable input is again x, but now x is the exponent and the constant, 3, is the base. Compared to power functions, the role of the base and exponent are reversed. Functions like this are called **exponential functions** because the exponent is the variable. The number $e \approx 2.718$ (learn more about e) will often be used as a base in caluculus.

Examples

1. Evaluate $2^{\frac{1}{3}} \cdot 4^{\frac{1}{3}}$
   Solution:

$$\begin{aligned} 2^{\frac{1}{3}} \cdot 4^{\frac{1}{3}} &= (2 \cdot 4)^{\frac{1}{3}} \\ &= 8^{1/3} \\ &= 2 \end{aligned}$$

   Remark: $8 = 2 \cdot 2 \cdot 2$, so $8^{1/3} = 2$.

2. Simplify $\left(\sqrt{3}\right)^{\frac{1}{2}}\left(\sqrt{12}\right)^{1/2}$ Remember: $x^{1/2} = \sqrt{x}$
   Solution:

$$\begin{aligned} \left(\sqrt{3}\right)^{\frac{1}{2}}\left(\sqrt{12}\right)^{\frac{1}{2}} &= \left(\sqrt{3} \cdot \sqrt{12}\right)^{\frac{1}{2}} \\ &= \left(\sqrt{3 \cdot 12}\right)^{\frac{1}{2}} \end{aligned}$$

$$= \left(\sqrt{36}\right)^{\frac{1}{2}}$$
$$= 6^{\frac{1}{2}} = \sqrt{6}$$

Practice

Use the Rules of Exponents to simplify the expressions.

1. $16^2 \cdot 16^{-1.75}$

2. $\frac{4^{4.2}}{4^{3.7}}$

3. $\left(25^{\frac{1}{8}}\right)^4$

4. $\left(\frac{2}{\sqrt{2}}\right)^4$

Find the domain and range of the following function

5. $f(x) = \frac{1}{2+e^x}$

Logarithms

Logarithms and exponentials are related. They are two different ways to express the relationship between numbers, a, y, and x. If

$$a^y = x \quad \text{then} \quad \log_a x = y.$$

These are equivalent statements and an oft used bridge between logarithms and exponentials. In the first statement, it's clear that y is an exponent. Thus, $\log_a x$ must also be an exponent since it is equal to y. Therefore, the rules for logarithms are quite similar to the rules of exponents discussed above.

Natural Logarithm and Base 10: When the base of the logarithm is either the number e or when the base is 10 the notation is simplified since these two bases are so commonly used.

$$\ln x = \log_e x \quad \text{and} \quad \log x = \log_{10} x$$

In calculus, we will favor the natural logarithm, $\ln x$, over other bases in order to avoid nasty constants popping up in our work.

Rules of Logarithms: If $b > 0$ and $x > 0$, the logarithm satisfies the following rules.

	Rule	**Example**
(1)	$\log_b(xy) = \log_b(x) + \log_b(y)$	$\ln(3x) = \ln 3 + \ln(x)$
(2)	$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$	$\ln\left(\frac{3}{x+1}\right) = \ln(3) - \ln(x+1)$
(3)	$\log_b(x^y) = y \log_b x$	$\ln(x^3) = 3 \ln x$

Additional properties (here we'll just focus on the natural logarithm)

(4) $\ln e^x = x, \quad x > 0$

(5) $e^{\ln x} = x, \quad x > 0$

(6) $\log_a x = \frac{\ln x}{\ln a}, \quad (a > 0, \ a \neq 1)$

Examples

1. Evaluate $\ln\left(\frac{1}{e^{-x}}\right)$

   Solution:

$$\ln\left(\frac{1}{e^{-x}}\right) = \ln 1 - \ln(e^{-x}) \quad \text{Rule (2) left to right}$$
$$= 0 - (-x)\ln e \quad \ln 1 = 0 \text{ and rule (3) left to right}$$
$$= x \ln e = x \quad \ln e^1 = 1 \text{ per property (4)}$$

2. Simplify $\ln(8x+4) - \ln 4$
   Solution:

$$\begin{aligned} \ln(8x+4) - \ln 4 &= \ln\big(4(2x+1)\big) - \ln 4 \\ &= \ln 4 + \ln(2x+1) - \ln 4 \\ &= \ln(2x+1) \end{aligned}$$

Practice

Simplify the following

1. $\ln(3x^2 - 9x)$

2. $\frac{1}{2}\ln(4t^4) - \ln(2)$

3. $e^{\ln(2)}$

4. $e^{-\ln(2)}$

5. $e^{\ln(x)-\ln(y)}$

6. Solve for y in terms of t, where $\ln(y) = 2t + 4$.

7. Solve for k, where $e^{2k} = 4$.