INITIAL ORBIT DETERMINATION THROUGH OPTICAL OBSERVATIONS

by

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Understanding the current and future position of the objects in earth orbit is important to national security and the safety of manned space flight. When a new object is sighted in orbit, the path of the orbit needs to be determined. Before the invention of radar this could only be done using the right ascension, declination angles and time. This method is still used to increase the accuracy of radar measurements. The use of optical observations instead of radar measurement was chosen because it is an important step to understand how orbiting objects can be measured from a rotating earth. Angle data is collected from the earth and converted into a coordinate system that is not rotating. The angle data does not include the distance from the earth.
to the satellite. Using three sets of measurements the orbit of the satellite can be determined. This was done for the Moon as well as the international space station. The distance from the center of the earth to the satellite is determined and compared to known values.
I would like to thank my adviser Dr. McNeil for his mentorship as I researched and wrote my thesis. In addition to my adviser I would like to thank my thesis committee Brother Oliphant and Brother Tonks for reviewing my work. I would also like to thank Brother Hansen for the time he spent with me in class and outside of class helping me become a better writer. Lastly I would like to thank my wife Gidianny For her support both financially and emotionally while I wrote my thesis.
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Chapter 1

Introduction and Background

1.1 Space Situational Awareness

The field of space situational awareness (SSA) is comprised of knowing what is in orbit around earth, where it is now located and where it will be located in the future. This is important to national security and the safety of manned space flight. Initial orbit determination is a computerized and mostly automated process. It is a problem that Laplace, Gauss, Newton and Euler all worked on without the aid of a computer. The method that Laplace used for orbit determination provides an understanding of the physics of orbit determination and thus provides a good entry point into the field of space situational awareness. A solid understanding of the Laplace method for orbit determination gives the background needed to understand other methods and enter the field of SSA.
1.2 Importance of Orbit Determination for Earth Satellites

In order to protect its assets in space the United States maintains a list of objects in earth orbit.[1] This list is used to calculate when objects such as the international space station and any other orbiting object are on a collision course. When two items are on a collision course one of them can either raise or lower its orbit in order to avoid the collision. Orbit determination is also important to national security. The United States tracks the satellites of foreign powers, ballistic missiles as well as identifying other objects in orbit.[1] When a new earth satellite is detected its orbit needs to be determined. One Method of determining the position and future orbit of a satellite is by using Laplace’s method for orbit determination.

1.3 Laplace’s Method for Orbit determination

This paper shows how Laplace’s method can be used to make an initial orbit determination. In order to gain a better understanding of the physics of orbit determination the method Laplace used will be introduced. This method uses instruments and techniques that would have been available to Laplace. MATLAB has been employed to take care of the repetitious arithmetic that is required for calculating dates, iteratively solving functions and least squares fitting. A computer is not needed to solve the problem; it was only used to save time. The method Laplace presented is demonstrated with angular data collected from the international space station as well
as for the Moon. The focus of the paper is to demonstrate how the Laplace method can be used to determine the position vector of an earth satellite. A knowledge of orbits is used to find the position and then three positions can be used to predict the orbit.
Chapter 2

Methods

Using the Laplace method of orbital determination to find the position vector for a satellite at a specific time requires three observations. Each of these three observations need to include three numbers, the elevation angle, the azimuthal angle and the universal time (UT1). Because the observations are taken in a spinning reference frame they need to be converted into a reference frame that is not spinning. This requires the observer to know the angle (local sidereal time) which the earth has spun since the observing station was aligned with the x axis of the non-rotating reference frame. In order to find the local sidereal time the Greenwich sidereal time must also be known. When converting from the topocentric coordinate system located on the surface of the earth into the geocentric coordinate system the non-spherical shape of the earth needs to be taken into account. Before calculating the position vector for the satellite an equation for a unit vector with respect to time needs to be formed. Once the unit vector and its derivatives have been calculated it can be used along
with the vector that points from the center of the earth to the observing station to calculate the position vector of the satellite.

2.1 Observation

The two fundamental tools for orbit determination are radar and optical telescopes. When used together these tools can be used to calculate a highly accurate orbit, accurate enough to calculate the position of the satellite at a future time. Radar data returns highly accurate velocity and distance information however it lacks angular resolution. Optical observations however return highly accurate angular resolution but poor velocity and distance measurements. The Laplace method uses optical observations to calculate distance measurements that are inherent to radar measurements.

The methods chosen to observe satellites in low earth orbit and for satellites in a high orbit were different. The Moon is in a high earth orbit so it moves slowly across the sky. Right ascension and declination angles for the Moon could be taken using a scope because of the slow relative movement between the observing station and the Moon. The Moon was centered in the scope and the right ascension was measured using a compass and the declination was measured using a protractor. The scope was home-built using a meter stick two pieces of card stock and a tripod. The apparatus gave measurements accurate to one degree for both right Ascension and declination. This data was within one degree when compared to the data from the Naval Observatory; however it did not contain the accuracy ($\pm 0.1^\circ$) needed to
calculate the orbital parameters. Because of this limitation the data from the Navel Observatory was used to find the position vector for the Moon.

**Figure 2.1** Scope for measuring the azimuthal and elevation angles of the Moon

Data for the international space station, which is a satellite in low earth orbit, was collected from the space station live streaming provided by satflare.[3] When the space station was in the sky at the observation location three data points were collected by taking screen shots of the tracking data. The only measurements used from the data provided by the tracking system were the azimuthal angle, elevation angle and the time. These are the three data points that would be available if the data was taken using an optical telescope.
2.2 Coordinate Systems

The coordinate system that measurements are taken in is referred to as the station coordinate system or the topocentric-horizon coordinate system. In the station coordinate system the z axis points to the zenith, the x axis points south towards the horizon and the y axis points east towards the horizon. This system with the origin located on the surface of the earth where the observations are made is often referred to as the SEZ coordinate system. Optical observations in this coordinate system are measured in azimuth and elevation angles. The azimuthal angle is measured eastward from the south and the elevation is the angle measured up from the horizon. In order to transform these measurements into an inertial reference frame recording an accurate time of observation is also important.

The coordinate system that the measurements are transformed into is the geocentric-equatorial coordinate system. In this system the x y plane is formed by the equatorial plane. The Z axis points north with the origin at the center of the earth. The x axis points in the direction of the vernal equinox and the y direction is 90 degrees counterclockwise from that in the equatorial plane. This coordinate system is referred to as the XYZ coordinate system. The transformation matrix used to transform from the SEZ coordinate system to the XYZ coordinate system is given in equation 2.1. In equation 2.1 \( \theta \) is the local sidereal time or the angle that separates the observation location and the vernal equinox and \( L \) is the latitude of the observing station.
2.3 Sidereal Time

In order to transform vectors in SEZ into IJK components the spin of the earth needs to be taken into account. To do this the angle $\theta$ between the vernal equinox and the longitude of the observation site needs to be calculated. This angle is called local sidereal time.

$$\theta = \theta_g + \lambda_E$$

The angle $\theta_g$ is the angle between the vernal equinox and Greenwich England at the time of observation. The angle $\lambda_E$ is the angle between Greenwich England and the observation location (i.e. geographic longitude).
2.3.1 Calculating Greenwich Mean Sidereal Time (GMST)

The angle between Greenwich England and the vernal equinox is calculated by counting how many times the earth has rotated with respect to the stars since a known time where Greenwich and the vernal equinox were aligned. Most formulas to calculate this use the Julian date which is the number of days since noon Greenwich mean time on January, 1st 4713 BC. [2]

\[ JD = 24553004.5 + \text{day of year} + \text{fraction of day from } 0^h \text{ UT} \]
2.4 Ellipsoid Earth Model

Once the Julian date has been calculated it can be used to calculate the GMST.

\[
\text{GMST} = 24110.54841 + 8640184.812866 \times \frac{\text{JD} - 2451545.0}{365.25}
\]

Because the GMST needs to be calculated for each optical or radar measurement a code to calculate it has been included in appendix A. [4]

2.3.2 Calculating Local Sidereal Time

Once the the Greenwich mean sidereal time is calculated and is in converted to degrees the local sidereal time can be obtained. This is done by adding the longitude of the observing station to the Greenwich mean sidereal time as seen in equation 2.3.

2.4 Ellipsoid Earth Model

The local sidereal time defines the angle between the vernal equinox and a plane that cuts through the center of the earth and contains the rotational axis. In order to define the stations coordinates a point on the plane must be defined. The observing station is on the surface of the earth and the surface of the earth is a certain distance above sea level. A point of the plane is defined using the latitude and elevations of the observing station. The earth is not however a perfect sphere, it has a slight pear shape.[1] To simplify the mathematics the cross-section of the earth can be modeled using an ellipse. Although not a perfect model an ellipse gives a good approximation. The eccentricity of the earth is \(e = 0.08182\).[1]
Using the fact that the line normal to an ellipse is given by $-dx/dz$ and the angle $L$ is the latitude the point on the ellipse can be calculated.

\begin{align}
x &= \left| \frac{a_e}{\sqrt{1 - e^2 \sin^2(L)}} \right| + \text{elevation} \cos L \tag{2.2} \\
z &= \left| \frac{a_e(1 - e^2)}{\sqrt{1 - e^2 \sin^2(L)}} \right| + \text{elevation} \sin L \tag{2.3}
\end{align}

The Vector from the center of the earth to the observing station $\vec{R}(t)$ defines the station coordinates.

$$\vec{R}(t) = x \cos \theta(t) \hat{I} + x \sin \theta(t) \hat{J} + z \hat{K}$$ \hspace{1cm} (2.4)
2.5 Calculating the Position Vectors

When an object that is in Earth orbit is observed with the naked eye from the Earth it is a difficult task to figure out how far away the satellite is. In order to accurately describe the position of and distance to a satellite a position vector is calculated. Each observation is converted into a unit vector that points toward the satellite (\( \hat{L} \)), these vectors are called the line of sight unit vectors. Once the line of sight unit vector with respect to time has been calculated it can be used to calculate the position vector (\( \vec{r} \)).

2.5.1 Line of Site Unit Vectors

The line of sight unit vector \( \hat{L}_i \) can be calculated for each measurement that is taken of the satellite. \( \hat{L}_i \) is a unit vector that points to the satellite. Like all unit vectors it has a magnitude of one and points in a certain direction.

\[
\hat{L}_i = \begin{bmatrix} \hat{L}_I \\ \hat{L}_J \\ \hat{L}_K \end{bmatrix} = \begin{bmatrix} \cos(El_i) & \cos(Az_i) \\ \cos(El_i) & \sin(Az_i) \\ \sin(El_i) \end{bmatrix}
\] (2.5)

In order to calculate the range and the range rate the first and second derivatives of \( \hat{L}_i \) with respect to time, \( \dot{\hat{L}}_i, \ddot{\hat{L}}_i \) need to be calculated.

There are two different ways to calculate \( \dot{\hat{L}}_i \) and \( \ddot{\hat{L}}_i \). The method chosen depends on the type of data that is available. If only three data points are available, which is often the case when a new satellite is discovered, \( \dot{\hat{L}}_i \) and \( \ddot{\hat{L}}_i \) will need to be numerically differentiated as shown below. [1]
If a larger number of observations are available then a least squares polynomial fit can be used to fit the data. Once a polynomial has been fit the first and second derivatives can easily be found. This method is more accurate but often there is not a lot of data available for initial orbit determination.

### 2.5.2 Position Vectors

The position vector can be defined as \( \vec{r} = \rho \hat{L} + \vec{R} \) as shown in figure 2.4, where \( \rho \) is the magnitude to the line of sight vector.
After $\vec{R}(t)$ and $\vec{L}(t)$ have been obtained $\vec{r} = \rho \hat{L} + \vec{R}$ can be differentiated twice with respect to time to obtain $\vec{r}'' = 2\dot{\rho} \vec{L} + \ddot{\rho} \hat{L} + \rho \vec{L}'' + \dddot{\vec{R}}$. $\vec{r}$ is the acceleration due to gravity and can be replaced with $-\mu \vec{r}$ giving equation 2.9

$$\dddot{L} \rho + 2 \ddot{L} \dot{\rho} + (\dddot{L} + \frac{\mu}{r^3} \vec{L}) \rho = - (\vec{r} + \mu \frac{\vec{R}}{r^3})$$  (2.9)

Cramer’s rule is then used to solve equation 2.9 for $\rho$. In order to solve for $\rho$ the determinate of the coefficients (equation 2.10) from equation 2.9 is used.

$$D = \begin{vmatrix} L_i & 2\dot{L}_i & \ddot{L}_i + \mu \frac{L_i}{r^3} \\ L_j & 2\dot{L}_j & \ddot{L}_j + \mu \frac{L_j}{r^3} \\ L_k & 2\dot{L}_k & \ddot{L}_k + \mu \frac{L_k}{r^3} \end{vmatrix}$$  (2.10)

Equation 2.10 can be simplified to equation 2.11.
\[ D = 2 \begin{vmatrix} L_i & \dot{L}_i & \ddot{L}_i \\ L_j & \dot{L}_j & \ddot{L}_j \\ L_k & \dot{L}_k & \ddot{L}_k \end{vmatrix} \] (2.11)

The result of applying Cramer’s rule to equation 2.11 is equation 2.12.

\[ D\rho = -2 \begin{vmatrix} L_i & \dot{L}_i & \ddot{R}_i \\ L_j & \dot{L}_j & \ddot{R}_j \\ L_k & \dot{L}_k & \ddot{R}_k \end{vmatrix} - \frac{2\mu}{r^3} \begin{vmatrix} L_i & \dot{L}_i & R_i \\ L_j & \dot{L}_j & R_j \\ L_k & \dot{L}_k & R_k \end{vmatrix} \] (2.12)

For simplification the first and second determinate of equation 2.12 will be replaced with \( D_1 \) and \( D_2 \). Solving equation 2.12 for \( \vec{\rho} \) yields equation 2.13.

\[ \vec{\rho} = -\frac{2D_1}{D} - \frac{2\mu D_2}{r^3 D} \] (2.13)

Having solved for \( \rho \) it can be substitute into \( \vec{r} = \rho \hat{L} + \vec{R} \) dotted with itself.

\[ r^2 = \vec{\rho}^2 + 2\rho \hat{L} \cdot \vec{R} + \vec{R}^2 \] (2.14)

The substitution of \( \rho \) into equation 2.14 yields the eighth order equation 2.17 which has been simplified by defining \( q \) and \( f \) as seen below.

\[ q = -\frac{2\mu D_2}{D} \] (2.15)

\[ f = -\frac{2D_1}{D} \] (2.16)

\[ 0 = r^6 f^2 + 2r^3 f q + q^2 + 2r^6 f \hat{L} \cdot \vec{R} + 2r^3 q \hat{L} \cdot \vec{R} + r^6 \vec{R}^2 - r^8 \] (2.17)
Equation 2.17 can be solved using Newtons iterative method. Having solved for $r$ in the case of the international space station as well as the Moon the results section will compare the calculated values with the known values.
Chapter 3

Results

3.1 Observations

The three observations for the international space station are seen in the three images below and the data that was used from these three images is shown in table 3.1.

Figure 3.1 ISS Data Point 1 [3]
Figure 3.2 ISS Data Point 2 [3]

![ISS Position and HD Camera FOV (Sunset in 33 min 50 s)](image)

Figure 3.3 ISS Data Point 3 [3]

![ISS Position and HD Camera FOV (Sunset in 31 min 32 s)](image)

Table 3.1 ISS Observations

<table>
<thead>
<tr>
<th>Observation #</th>
<th>Latitude</th>
<th>Longitude</th>
<th>Year</th>
<th>Month</th>
<th>Day</th>
<th>Hour</th>
<th>Minute</th>
<th>Second</th>
<th>Az angle</th>
<th>El angle</th>
<th>Elevation R°</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>43.813°</td>
<td>-111.809°</td>
<td>2016</td>
<td>4</td>
<td>26</td>
<td>17</td>
<td>21</td>
<td>1</td>
<td>238.5°</td>
<td>17.4°</td>
<td>0.0002325</td>
</tr>
<tr>
<td>2</td>
<td>43.813°</td>
<td>-111.809°</td>
<td>2016</td>
<td>4</td>
<td>26</td>
<td>17</td>
<td>23</td>
<td>24</td>
<td>2.9°</td>
<td>86.8°</td>
<td>0.0002325</td>
</tr>
<tr>
<td>3</td>
<td>43.813°</td>
<td>-111.809°</td>
<td>2016</td>
<td>4</td>
<td>26</td>
<td>17</td>
<td>25</td>
<td>42</td>
<td>57.3°</td>
<td>17.8°</td>
<td>0.0002325</td>
</tr>
</tbody>
</table>
3.2 Sidereal Time Angles

Table 3.2 contains the data used from the Naval Observatory. It was used to calculate the distance to the moon at the time of observation.

<table>
<thead>
<tr>
<th>Observation #</th>
<th>Latitude</th>
<th>Longitude</th>
<th>Year</th>
<th>Month</th>
<th>Day</th>
<th>Hour</th>
<th>Minute</th>
<th>Second</th>
<th>Az angle</th>
<th>El angle</th>
<th>Elevation R(\oplus)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>43.813°</td>
<td>-111.809°</td>
<td>2016</td>
<td>3</td>
<td>19</td>
<td>10</td>
<td>0</td>
<td>96.1°</td>
<td>6.6°</td>
<td>0.0002325</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>43.813°</td>
<td>-111.809°</td>
<td>2016</td>
<td>3</td>
<td>19</td>
<td>17</td>
<td>40</td>
<td>141.6°</td>
<td>39.5°</td>
<td>0.0002325</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>43.813°</td>
<td>-111.809°</td>
<td>2016</td>
<td>3</td>
<td>19</td>
<td>23</td>
<td>50</td>
<td>251.5°</td>
<td>18.8°</td>
<td>0.0002325</td>
<td></td>
</tr>
</tbody>
</table>

3.2 Sidereal Time Angles

Each observation was taken at a different time so each observation has a different local sidereal time angle. Each observation and the corresponding local sidereal time angle is listed in table 3.3. These angles were calculate using the local sidereal time code in appendix A.

<table>
<thead>
<tr>
<th>Observation #</th>
<th>(\theta) ISS</th>
<th>(\theta) Moon</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.584°</td>
<td>278.3°</td>
</tr>
<tr>
<td>2</td>
<td>4.181°</td>
<td>330.9°</td>
</tr>
<tr>
<td>3</td>
<td>4.758°</td>
<td>63.64°</td>
</tr>
</tbody>
</table>

3.3 Station Coordinates With Respect to Time

Using the equatorial radius \(a_e\) of 6356.785 km, eccentricity of the Earth \(e\) of 0.08182, station elevation of 0.00023251 earth radii and station latitude of 43.813°
the effects of the ellipsoid Earth were calculated.

\[
x = 0.7229 \\
z = 0.6889
\]

When the \(x\) and \(z\) values are entered into equation 2.4 the station coordinates (\(\vec{R}\)) with respect to local sidereal time is known.

\[
\vec{R}(t) = 0.7229 \cos \theta(t) \hat{I} + 0.7229 \sin \theta(t) \hat{J} + 0.6889 \hat{K}
\]

### 3.4 Line of Sight Unit Vector With Respect to Time

Using the information collected so far the equation for the line of sight unit vector with respect to time can be formulated using equation 2.6.

\[
\hat{L}_{ISS}(\theta) = 0.6798 \hat{x} + 0.0525 \hat{Y} + 0.7314 \hat{z}
\]

\[
\hat{L}_{Moon}(\theta) = 1.000 \hat{x} - 0.0039 \hat{Y} + 0.0064 \hat{z}
\]

### 3.5 Results for the Moon

The Moon was calculated to be 298 earth radii from the center of the earth at the time of the middle observation. This is over five times the average distance from the earth to the moon. Although the calculated and actual distance to the moon are not the same, we were able to tell the the moon is far from the earth.
3.6 Results for a Low Earth Orbit Satellite

The calculated distance from the center of the earth to the international space station was 1.145 earth radii. The actual distance from the center of the earth is 1.0638 earth radii. The calculated distance to the ISS was 1.08 times the actual distance.

3.7 Error Analysis

Matlab was used to calculate the error in the magnitude of the position vector resulting from measurement error. The program that calculates the magnitude of the position vector was run one million times to create a distribution of possible answers, each time randomly selecting input values that are within the range of the error for the measurements as shown in table 3.4. The error parameters shown in the table are for the International space station. The only error that is different for the Moon is the time is ±.5 minutes.

<table>
<thead>
<tr>
<th>Measurement</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>El</td>
<td>±0.1°</td>
</tr>
<tr>
<td>Az</td>
<td>±0.1°</td>
</tr>
<tr>
<td>Elevation</td>
<td>±10⁻⁸ Earth radii</td>
</tr>
<tr>
<td>Time</td>
<td>±0.5 s</td>
</tr>
<tr>
<td>Latitude</td>
<td>± 0.001 °</td>
</tr>
<tr>
<td>Longitude</td>
<td>± 0.001 °</td>
</tr>
</tbody>
</table>

Figure 3.5 shows the magnitude of the position vector vs. the number of instances for the International Space Station. From the data used to make the distribution in
Figure 3.4 Error from ISS Measurements

For the moon the calculated value for the distance to the moon is five times the actual. The error calculations show that the actual distance to the moon lies within
3.8 Comparison

The code returns distances that are larger than the actual distances. There are a number of possible causes to the exaggerated distances. One reason for the error in the distances could be the section of the orbit that was used was too small. For the calculations in this paper, less than ten percent of the orbit was used. Using greater than ten percent of the orbit would decrease the error and increase the accuracy of the results. The code is run with three data points. Altering the code to include
more than three data points would also decrease the error and increase the accuracy of the results.

Although the results do not exactly match the actual data the program does indicate whether a satellite is close to or far away from the earth. We guessed that the larger the section of an orbit used in a calculation the more accurate the magnitude of the position vector will be. The data for the Moon spanned 9 hours comprising only one percent of the Moon’s orbit. The data taken from the international space station spans 4.68 minutes this is 5 percent of the 92.6 minute period. As we guessed the observations the spanned the greater percentage of the orbit were more accurate.
Chapter 4

Conclusion

4.1 Outcome

Laplace’s method of orbit determination was used to calculate the vector to an orbiting object using only angles, time and the observation location on the Earth. The magnitude of the position vector was compared to the altitude of the satellite. It was shown that Laplace’s method for orbit determination can be used to calculate the distance to an object in orbit around the Earth.

4.2 Further Research

There are many ways to improve what was done in this paper. A few of the methods to improve the accuracy or the results are listed below. When secondary data from a down range observation site is available the data can be used to improve the accuracy of the orbit. The data is used to do a differential correction to the satellites orbit.
This could be accomplished for the international space station without changing location. The data for the space station could be collected the same as stated in section 2.1 and then a down range location could be selected on the computer and a second set of data could be collected for that location.

In addition to the Laplace method for orbit determination that was used above there is a second common method that was invented by Gauss. The Gauss method could be used as well to compare results.

4.2.1 Calculating Orbital Elements

In addition to improving the results the position vector the orbital elements can be calculated. After the three coplanar position vectors have been obtained the orbital elements can be calculated. The vector algebra used to calculate the orbital elements is shown in appendix B.

4.2.2 Polynomial Fit

When a large number of right ascension and declination data points are available an equation with respect to time for the line of sight vectors ($\vec{L}_i$) can be fit to the data. This can be accurately done using a least squares polynomial fit. This is a more accurate method than using the equations 2.6 through 2.8 for $\vec{L}_i$. In figure 4.1 the right ascension and declination data for the Moon with the polynomial fits is graphed. This process returns a polynomial that describes how the right ascension and declination change with time. Putting this information into equation B.11 gives an equation for the line of sight unit vectors with respect to time, the derivative of
In order to obtain an equation for the station coordinates with respect to time, a sine function was fit to the data. This was done by minimizing chi squared. An initial guess for the parameters of the sine function was made and chi squared was calculated. A MATLAB program then adjusted the parameters using the linear least squares fitting method until chi squared was at a minimum.
Bibliography


Appendix A

MATLAB Code

% The section of code below finds the magnitude of the position vector.

close all;
clear all;

% Observation data
year=2016;
month=4;
day=26;
hour=17;
E1=[17.4,86.8,17.8];
Az=[238.5,2.9,57.3];
minute=[21,23,25];
secound=[1,24,42];
latitude=43.813; % Of observing station
longitude=-111.809; % Of observing station
elevation=0.00023251274; % Of observing station

ae=1; % average earth radii in average earth radii
ec=0.08182; % Eccentricity of the surface of the earth
sid=86164.09054; % Seconds in a day.
% Calculates the x value for the station coordinates based off of the
% ellipsoid Earth model.
x=abs(ae/sqrt(1-ec^2*sind(latitude)^2)+elevation)*cosd(latitude)
%
theta=[localSiderealTime(year,month,day,hour,minute(1),secound(1),...%longitude),localSiderealTime(year,month,day,hour,minute(2),...%secound(2),longitude),localSiderealTime(year,month,day,hour,...%minute(3),secound(3),longitude)];
theta0=localSiderealTime(year,month,day,0,0,0,longitude);

% DRot1,2&3 are the rotation matrixes that will transform the line of sight
% unit vectors (L1,2&3) from the SEZ to the XYZ reference frame.
DRot1 = [sind(latitude)*cos(theta(1)),-sin(theta(1)),cosd(latitude)*...%cos(theta(1));sind(latitude)*sin(theta(1)),cos(theta(1));...%cosd(latitude)*sin(theta(1));-cosd(latitude),0,sind(latitude)];
DRot2 = [sind(latitude)*cos(theta(2)),-sin(theta(2)),cosd(latitude)*...%cos(theta(2));sind(latitude)*sin(theta(2)),cos(theta(2));...%cosd(latitude)*sin(theta(2));-cosd(latitude),0,sind(latitude)];
DRot3 = [sind(latitude)*cos(theta(3)),-sin(theta(3)),cosd(latitude)...%*cos(theta(3));sind(latitude)*sin(theta(3)),cos(theta(3));...
\texttt{cosd(latitude) \times \sin(theta(3)); -cosd(latitude), 0, \sin(latitude)];}

% Converts each time from hour minute and second to seconds.
\texttt{t1=hour*60*60+minute(1)*60+secound(1);}
\texttt{t2=hour*60*60+minute(2)*60+secound(2);}
\texttt{t3=hour*60*60+minute(3)*60+secound(3);}

% Select the time when you want to calculate the position vector.
\texttt{t=t2;}

% Line of sight unit vector for each of the three observations
\texttt{L1=DRot1*[cos(El(1)*pi()/180)*cos(pi()-Az(1)*pi()/180);...}
\texttt{cos(El(1)*pi()/180)*sin(pi()-Az(1)*pi()/180); sin(El(1)*pi()/180)];}
\texttt{L2=DRot2*[cos(El(2)*pi()/180)*cos(pi()-Az(2)*pi()/180);...}
\texttt{cos(El(2)*pi()/180)*sin(pi()-Az(2)*pi()/180); sin(El(2)*pi()/180)];}
\texttt{L3=DRot3*[cos(El(3)*pi()/180)*cos(pi()-Az(3)*pi()/180);...}
\texttt{cos(El(3)*pi()/180)*sin(pi()-Az(3)*pi()/180); sin(El(3)*pi()/180)];}

% Line of sight unit vector with respect to time and
% its first and second derivatives

% Station coordinates R and the second derivative of R, Rdd
R=[x*cos(theta0+t/sid*2*pi());x*sin(theta0+t/sid*2*pi());...
  0.688939127608169];
Rdd=[-5.2885*10^-9*x*cos(theta0+t/sid*2*pi());-5.2885*10^-9*x*...
   sin(theta0+t/sid*2*pi());0];

% Constants in the eighth order polynomial
D1=det([L,Ld,Rdd]);
D2=det([L,Ld,R]);
D=2*det([L,Ld,Ldd]);
mu=1.53623*10^-6; % Gravitational parameter in earth radii^3/second^2.
f=-2*D1/D;
q=-2*mu*D2/D;
ddd=dot(L,R);
su=sum(R.^2);

r=2; % Guess where to start the iterative solving method
z=zeros(1,20); % Array initialization to track the iterative solving loop.

% The for loop below uses Newtons iterative solving method to solve the
% eighth order polynomial for the magnitude of the position vector.
for i=1:20 % iterative solving loop
  rplus=r-(f^2*r^6+2*f*q*r^3+q^2+2*f*dot(L,R)*r^6+2*q*dot(L,R)*r^3-r^8+...
     sum(R.^2)*r^6)/(6*f^2*r^5+6*f*q*r^2+12*f*dot(L,R)*r^5+6*q*dot(L,R)...
     *r^2-8*r^7+6*sum(R.^2)*r^5);
  r=rplus;
  z(1,i)=r;
end
% Display the magnitude of the position vector.

\[ r \]

% Uncomment the plot below to see if the position vector has converged.

% plot(z

% Turns the Greenwich Mean Siderial time into local siderial time

function \[ \theta \] = localSiderealTime(year, month, day, hour, minute, second, longitude)
% Calls the Greenwich Mean Sidereal Time function
[\( G_h \), \( G_m \), \( G_s \)] = greenwichMeanSiderealTime(year, month, day, hour, minute, second);
% turns the longitude into hours minutes and seconds
localHours = longitude / 360 * 24;
plusLh = floor(localHours);
plusLm = floor((localHours - plusLh) * 60);
plusLs = (localHours - plusLh - plusLm / 60) * 60 * 60;
% adds the longitude to the Greenwich Mean Siderial time
Lh = \( G_h \) + plusLh;
Lm = \( G_m \) + plusLm;
Ls = \( G_s \) + plusLs;

% the if statements make sure that there are the correct number of hours
% minutes and seconds in the day
if Ls >= 60;
    Ls = Ls - 60;
    Lm = Lm + 1;
end
if Lm >= 60
    Lh=Lh+1;
    Lm=Lm-60;
end

if Lh >= 24
    Lh=Lh-24;
end

Lh;
Lm;
Ls;

% gives the local sidereal time in radians and degrees
theta=(Lh+Lm/60+Ls/60/60)/24*2*pi(); % radians
thetad=theta*180/(2*pi()); % degrees

% This function calculates the Greenwich Mean Sidereal Time
function [Gh,Gm,Gs]=greenwichMeanSiderealTime(year,month,day,hour,...
    minute,secound)
    % Calls the Julian Date funcion
    JD=julianDate(year,month,day,hour,minute,secound);
    % Brings the greenwich maridian inline with the vernal equinox
    D=JD-2451545.0;
    GMSTNotReduced = 18.697374558 + 24.06570982441908*D;
    GMST = mod(GMSTNotReduced,24);
    % turns the GMST into the hour minute and secound format
    Gh=floor(GMST);
Gm = floor((GMST - Gh) * 60);
Gs = (GMST - Gh - (Gm / 60)) * 60 * 60;
Appendix B

Calculating Orbital Elements

First the $\vec{D}$, $\vec{N}$ and $\vec{S}$ vectors are calculated.

$$\vec{D} = \vec{r}_1 \times \vec{r}_2 + \vec{r}_2 \times \vec{r}_3 + \vec{r}_3 \times \vec{r}_1$$  \hspace{1cm} (B.1)

$$\vec{N} = r_3 \vec{r}_1 \times \vec{r}_2 + r_1 \vec{r}_2 \times \vec{r}_3 + r_2 \vec{r}_3 \times \vec{r}_1$$  \hspace{1cm} (B.2)

$$\vec{S} = (r_2 - r_3)\vec{r}_1 + (r_3 - r_1)\vec{r}_2 + (r_1 - r_2)\vec{r}_3$$  \hspace{1cm} (B.3)

With the $\vec{D}$, $\vec{N}$ and $\vec{S}$ vectors calculated the orbital elements can calculated. In order for $\vec{D}$, $\vec{N}$ and $\vec{S}$ to describe a possible orbit $\vec{D}$ and $\vec{N}$ can not be equal to zero and $\vec{D} \cdot \vec{N}$ must be greater than zero.

$$e = \frac{S}{D}, \text{ eccentricity}$$  \hspace{1cm} (B.4)

$$\vec{Q} = \frac{\vec{S}}{S}$$  \hspace{1cm} (B.5)

$$\vec{W} = \frac{\vec{N}}{N}$$  \hspace{1cm} (B.6)

$$\vec{P} = \vec{Q} \times \vec{W}, \text{ semi-latus rectum}$$  \hspace{1cm} (B.7)
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\[ a = \frac{P}{1 - e^2}, \text{semi-major axis} \]  \hspace{1cm} (B.8)

\[ \text{period} = 2\pi \sqrt{\frac{a^3}{mu}} \]  \hspace{1cm} (B.9)

\[ \vec{B} = \vec{D} \times \vec{r} \]  \hspace{1cm} (B.10)

\[ L = \frac{1}{\sqrt{DN}} \]  \hspace{1cm} (B.11)

\[ \vec{V} = \frac{L}{r} \vec{B} + L \vec{S}, \text{velocity} \]  \hspace{1cm} (B.12)

From these orbital elements any other orbital element can be calculated.